ISRAEL JOURNAL OF MATHEMATICS, Vol. 70, No. 1, 1990

A GENERALIZATION OF THE LOOMAN–MENCHOFF THEOREM

BY

N. V. RAO

Department of Mathematics, University of Toledo, Toledo, OH 43606, USA

ABSTRACT

In this paper we give a generalization of the classical Looman–Menchoff theorem: *If f is a complex-valued continuous function of a complex variable in a domain G, f has partial derivatives f_x and f_y everywhere in G and the Cauchy Riemann equations $f_x + if_y = 0$ are satisfied almost everywhere, then f is holomorphic in G.* From our generalization of this theorem, we deduce a theroem of Sindalovskii [9] as a corollary and also answer some of the questions raised in [9]. We note in this context that, as far as we know, Sindalovskii's result is the best published to date in this area.

1. Introduction

The classical Cauchy–Goursat theorem states: *If f is a complex-valued function of a complex variable $z = x + iy$ defined on a domain D and*

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists for all z in D, then f is holomorphic in D.

Ever since Goursat [3] proved this theorem in 1900, there have been various improvements and generalizations. In fact Pompeiu [7] in 1905 noted that Goursat's theorem would remain valid if one were to assume only that $f'(z)$ existed only almost everywhere in D and

$$\limsup_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

is bounded in D.

Received June 21, 1989 and in revised form November 13, 1989

But the most dramatic was the announcement by Montel [6] in 1913 without proof that *if f is bounded in D; f_x, f_y exist everywhere and $f_x + if_y = 0$ almost everywhere in D, then f is holomorphic in D.* We should note that f is not assumed to be continuous.

However, in 1923 Looman [4] proved Montel's announcement replacing boundedness of f by the stronger property, continuity of f. Apparently his proof contained a gap which was fixed by Menchoff. But Menchoff did not publish his proof. We see for the first time a complete proof of this theorem and also the attribution Looman–Menchoff in Saks [8]. See also Menchoff [5].

In 1941, Tolstoff [10] had established by giving a counter-example that the considerations on which Montel based his announcement were false. Further, in 1942, he [11] gave a correct proof for Montel's announcement of 1913. After Tolstoff, many have asked (see [9] for details) whether boundedness of f could be replaced by the L^1-integrability of f. This question was answered in the affirmative by Sindalovskii [9] and, in the same paper, he has asked the question whether it might not be enough to assume that 'f belongs to $L^p(D)$ for some $p > 0$'.

Let us recall the main theorem of Sindalovskii:

Suppose f is a complex-valued function of a complex variable z in a domain G. Further assume the following:

(a) *f is separately continuous in x, y where $x + iy = z$ in G,*

(b) *first order partial derivatives f_x, f_y exist everywhere in G except on a set $E = \bigcup_{n=1}^{\infty} E_n$ where each E_n is closed and of finite Hausdorff length,*

(c) *Cauchy–Riemann equations $f_x + if_y = 0$ are satisfied almost everywhere in G,*

(d) *and f belongs to the Lebesgue space $L^1(G)$.*

Then f is holomorphic in G.

There was an exceptional set allowed both for the Looman–Menchoff theorem and the Tolstoff theorem except that E is assumed to be countable. And in 1964, Fesq [2] while generalizing a version of Green's theorem due to P. Cohen also improved the exceptional set in Tolstoff's theorem to the form stated in (b).

We rephrase Sindalovskii's question as follows: Can (d) be replaced by

(d′) f belongs to $L^p(G)$ for some $p > 0$.

Let K denote any one of the four closed unit squares with a vertex at the origin in the complex plane and $\tilde{K}$ the interior of the set obtained by joining to

K the two adjacent unit squares. Let $K(r, z)$ denote the set $z + rK$ and similarly $\tilde{K}(r, z)$ denote $z + r\tilde{K}$. Let $\tilde{M}(r, K, z)$ denote $\max|f|$ on the two sides of $K(r, z)$ that do not contain z.

In this paper we replace (d) with

$$\text{(d}''\text{)} \qquad \log \tilde{M}(r, K, z) \leqq o(r^{-2}) \quad \text{at every } z \text{ in } G$$

where f is found to be holomorphic in $\tilde{K}(\delta, z)$ for some $\delta > 0$ however small.

We will show that (d′) implies (d″) which proves Sindalovskii's result and also answers his question in the affirmative.

Also, we construct an example of a function f which satisfies (a), (b) and (c), but fails to satisfy (d″) only at one point, thus establishing that our result is the best possible. Indeed f is holomorphic everywhere except at the origin and further that

$$\int_{\partial R} f(z)dz = 0$$

for every rectangle R with sides parallel to the axes.

For more detailed references regarding this problem, refer to Saks [8] and Sindalovskii [9].

2. The main theorem

We shall assume the following theorem obtained by Fesq [2] and hereafter refer to it as TF (Tolstoff–Fesq).

TF. *Let f be a complex-valued function of the complex variable $z = x + iy$ in a domain G. Assume that f satisfies* (a), (b), (c) *of the introduction and f is bounded in G. Then f is holomorphic in G.*

Also we recall a version of the classical Phragmén–Lindelof theorem (e.g. refer to Boas [1]) and refer to it as PL in the sequel.

PL. *Suppose that f is holomorphic on the square*

$$D = \{(x, y) : 0 \leqq x \leqq R, 0 \leqq y \leqq R\}$$

except at $(0, 0)$ *where it need not even be bounded. Further assume that*

$$|f(\zeta)| \leqq M \qquad \text{for all } \zeta \in \partial D \setminus \{0\}$$

and $\log \tilde{M}(r) \leqq o(r^{-2})$ *where* $\tilde{M}(r) = \sup|f|$ *on the lines $x = r$ or $y = r$ in D. Then $|f| \leqq M$ on all of D.*

This is slightly different from the usual versions of PL but can be deduced immediately from them.

THEOREM 2.1. *Let f satisfy* (a), (b), (c) *and* (d″) *of the introduction in a domain G of the complex plane. Then f is holomorphic in G.*

PROOF. Let $C_{m,n}$ be the set of points z in G such that

$$|f(z+h)-f(z)| \leqq m \qquad \text{for } |h| \leqq 1/n,$$

h real or purely imaginary ("ropi" from now on). We claim that

$$\bar{C}_{m,n} \subset C_{2m,n} \qquad \text{for every } m \text{ and } n$$

where $\bar{C}_{m,n}$ is the closure of $C_{m,n}$. Suppose z_k is a sequence of points in $C_{m,n}$ such that $\{z_k\} \to z_0$ as $k \to \infty$. Let $z_k = z_0 + \xi_k + i\eta_k$, ξ_k, η_k real. By hypothesis

$$|f(z_0+\xi_k+i\eta_k+h)-f(z_0+\xi_k+i\eta_k)| \leqq m \tag{2.2}$$

for $|h| \leqq 1/n$ and "ropi". Fix h' real so that $|h'| < 1/n$. Then, for sufficiently large k, $|\xi_k| + |h'| \leqq 1/n$ and so setting $h = h' - \xi_k$ in (2.2), we have

$$|f(z_0+h'+i\eta_k)-f(z_0+\xi_k+i\eta_k)| \leqq m \tag{2.3}$$

for all large k. But by (a), i.e. linear continuity,

$$f(z_0+h'+i\eta_k) \to f(z_0+h') \qquad \text{as } k \to \infty$$

and so all the limit points l of the sequence $\{f(z_k)\}$ satisfy

$$|f(z_0+h')-l| \leqq m.$$

This shows that $\{f(z_k)\}$ is a bounded sequence and therefore we may assume by going to a subsequence if necessary that

$$\{f(z_k)\} \to l \qquad \text{as } k \to \infty.$$

Hence we have

$$|f(z_0+h')-l| \leqq m \qquad \text{for } |h'| \leqq 1/n, \quad h' \text{ real.} \tag{2.4}$$

Therefore $|f(z_0+h')-f(z_0)| \leqq 2m$ for $|h'| \leqq 1/n$ and h' real. A similar argument will work for h' purely imaginary. Thus we have

$$\overline{C}_{m,n} \subset C_{2m,n}. \tag{2.5}$$

Because of linear continuity, we have

$$G = \bigcup_{n=1}^{\infty} C_{1,n}. \tag{2.6}$$

Let F be the set of points of G where f is not bounded, i.e. given any integer N and any neighbourhood V of a point z_1 in F, there exists a z in V such that $|f(z)| > N$. Clearly F is a closed subset of G.

We shall show that F is empty. But first, we claim

$$F^0 = \varnothing .$$

If not, since $F \subset \bigcup_{n=1}^{\infty} \overline{C}_{1,n}$, by the Baire category theorem there exists a square Q (squares in here have sides parallel to the axes) such that

$$Q^0 \subset F \subset \overline{C}_{1,n} \qquad \text{for some } n. \tag{2.7}$$

Let $z \in Q^0$ be the center of a square S centered at z and of size $\delta < 1/n$ and contained in $\overline{C}_{1,n}$. Suppose that $z_1, z_2 \in S$. Then $z_2 = z_1 + \xi + i\eta$, ξ, η real and $|\xi|, |\eta| < 1/n$. Since $z_1, z_2 \in C_{1,n}$ and $\overline{C}_{1,n} \subset C_{2,n}$, we have

$$|f(z_1) - f(z_1 + i\eta)| \leqq 2, \qquad |f(z_1 + i\eta + \xi) - f(z_1 + i\eta)| \leqq 2$$

and so

$$|f(z_1) - f(z_2)| \leqq 4.$$

Consequently f is bounded on S and z cannot belong to F and hence a contradiction.

Therefore

$$F^0 = \varnothing . \tag{2.8}$$

Once again, using the Baire category theorem, we find that there exists a square $Q \subset G$ such that

$$\varnothing \neq F \cap Q^0 \subset \overline{C}_{1,n} \subset C_{2,n} \qquad \text{for some } n. \tag{2.9}$$

Let $z \in F \cap Q^0$ and S be a square contained in Q^0 centered at z, of size $\delta < 1/n$. We say that

(2.10) "z_1 sees z_2" if $z_1 - z_2$ is "ropi",

(2.11) "F sees z" if some point of F sees z.

Thus in the square S, if a point z_1 of F sees z_2, then $|f(z_2) - f(z_1)| \leqq 2$ since

$z_1 \in C_{2,n}$, $|z_2 - z_1| < 1/n$ and $z_2 - z_1$ is "ropi". Similarly, if $z_1, z_2 \in F \cap S$, since they both see a ζ in S we obtain

$$|f(z_1) - f(\zeta)| \leqq 2, \quad |f(z_2) - f(\zeta)| \leqq 2 \quad \text{and} \quad |f(z_1) - f(z_2)| \leqq 4.$$

Therefore

$$(2.12) \qquad |f(z) - f(z_1)| \leqq 4 \qquad \text{for every } z_1 \text{ in } F \cap S.$$

Consequently any point ζ in S seen by $F \cap S$ satisfies

$$(2.13) \qquad |f(\zeta) - f(z)| \leqq 6 \quad \text{and} \quad |f(\zeta)| \leqq |f(z)| + 6.$$

Now we claim that if a point ζ_1 of F belongs to any open quadrant Q_1 of S (see Fig. I), then the quadrant Q_2 determined by z and ζ_1 is devoid of all points of F. If all points of Q_2 are seen by $F \cap S$, then by (2.13)

$$|f| \leqq |f(z)| + 6 \text{ in } Q_2 \quad \text{and} \quad F \cap Q_2 = \varnothing.$$

Suppose ζ is a point of Q_2 that is not seen by $F \cap S$. Then there is an open cross X bounded by lines l_1, l_2, l_3, l_4 (see Fig. II) containing ζ in its 'heart' H. All the lines l_1, l_2, l_3, l_4 must contain a point of $F \cap S$. If not, one can move l_1 up, l_2 down, l_3 to the left and l_4 to the right until they do so. Hence on l_1, l_2, l_3, l_4,

$$(2.14) \qquad |f| \leqq |f(z)| + 6.$$

Since $F \cap X = \varnothing$ and f is bounded at each point of X and so by TF, f is holomorphic in X. Now at each corner of H, PL can be applied to obtain

$$(2.15) \qquad |f| \leqq |f(z)| + 6 \qquad \text{in } H.$$

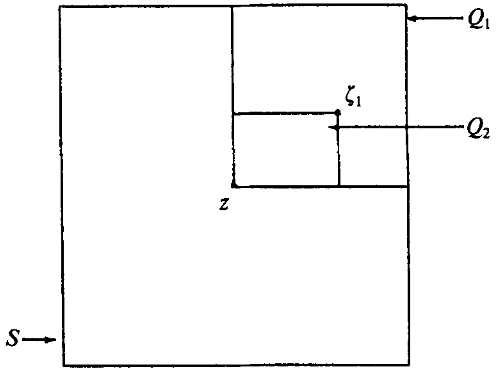

Figure I

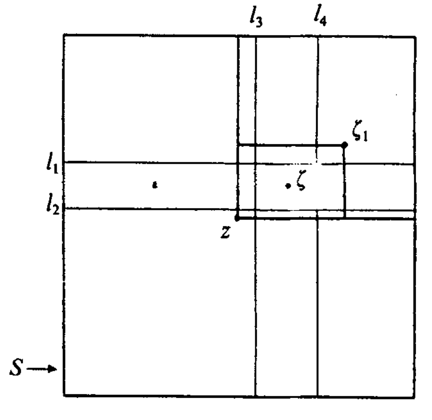

Figure II

Now (2.13) and (2.15) give us

$$|f| \leqq |f(z)| + 6 \text{ in the subquadrant } Q_2.$$

Thus $F \cap Q_2 = \varnothing$.

To sum up at this point, in each quadrant Q of S, there is a subquadrant with a vertex at z and free of points of F. This means that we may assume that

(2.16) $F \cap S$ is located only on the vertical or horizontal through z.

Now we claim that

(2.17) there cannot be more than two points of $F \cap S$ in any direction from z.

Suppose not. Say z_1, z_2, z_3 are three points of $F \cap S$ to the right of z (see Fig. III). First, there must exist an open interval (λ, μ) in $[z_1, z_3]$ not containing any points of F whereas $\lambda, \mu \in F$. If not, then all points in the vertical strip R' based on $[z_1, z_3]$ are seen by $F \cap S$ and so on R',

$$|f| \leqq |f(z)| + 4,$$

i.e. f is bounded on R' and hence at z_2, a contradiction.

Let R denote the vertical strip in S based on (λ, μ). Now f is bounded at each point of $\overline{R}$ except at λ and μ and so, by TF, f is holomorphic on R except at λ and μ. Now let $M = \max_{\partial S} |f|$. On the vertical boundary of R, $|f| \leqq |f(z)| + 4$ by (2.12) and since λ, μ belong to F. Again by PL, we have

(2.18) $$|f| \leqq \max(M, |f(z)| + 4) \qquad \text{on } R.$$

Thus on all vertical strips based on open intervals (λ, μ) of $[z_1, z_3]$ devoid of

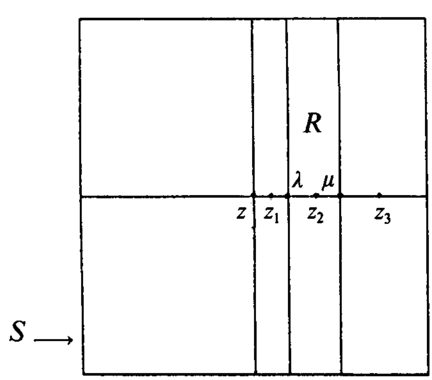

Figure III

points of F, $|f| \leqq \max(M, |f(z)| + 4)$ and all vertical lines through the points of F, $|f| \leqq |f(z)| + 4$. Consequently, in the vertical strip based on $[z_1, z_3]$, $|f| \leqq \max(M, |f(z)| + 4)$, i.e. f is bounded at z_2, a contradiction. This proves (2.17) and so z is an isolated point of F. By shrinking S some more if necessary, we can assume that $F \cap S = \{z\}$. We can again use PL at z in each quadrant and obtain

$$|f| \leqq \max(M, |f(z)| + 2)$$

in all S, i.e. f is bounded at z, a contradiction.

The theorem is established. □

3. Derivation of Sindalovskii's theorem

We recall Sindalovskii's theorem: If a function f satisfies conditions (a), (b), (c) and (d) of Section 1 on a domain G, then f is holomorphic in G.

In the same paper [9], Sindalovskii raised the question whether the following stronger theorem is true: If a function f satisfies the conditions (a), (b), (c) and (d′) of Section 1, then f is holomorphic in G.

In the sequel, we shall answer this question affirmatively. Let us assume that f satisfies (a), (b), (c) and (d′) over a domain G. Therefore we have

$$\int\int_G |f|^p = C < \infty \qquad \text{for some } p > 0.$$

Let z be any point of G for which there exists a $\delta > 0$ and a unit square K with vertex at the origin such that f is holomorphic on $\tilde{K}(\delta, z) = z + \delta\tilde{K}$.

Let $0 < r < \delta/2$ and ζ_0 be any point on any one of the two sides of $K(r, z)$ that do not contain z. Then the open disc $D(\xi_0, r)$ with center at ζ_0 and radius r is contained in $\tilde{K}$.

Because f is holomorphic in $D(\zeta_0, r)$ and $|f|^p$ is subharmonic there, we have

$$\pi r^2 |f(\zeta_0)|^p \leqq \int\int_{D(\zeta_0, r)} |f|^p dxdy \leqq C.$$

Thus

$$|f(\zeta_0)| \leqq \left(\frac{C}{\pi}\right)^{1/p} r^{-2/p}$$

and

$$\tilde{M}(r, K, z) \leqq \left(\frac{C}{\pi}\right)^{1/p} r^{-2/p};$$

we have

$$\log \tilde{M}(r, K, z) \leqq \frac{1}{p} \log \left(\frac{C}{\pi}\right) - \frac{2}{p} \log r \leqq o(r^{-2})$$

which is exactly the condition (d″).

Therefore f satisfies (a), (b), (c) and (d″) on the domain G. By Theorem 2.1, it follows that f is holomorphic in G. We may remark that (d′) need be satisfied only locally in G and p could depend on the location.

4. Counter-example

Let $f(z) = z^2 e^{iz^{-2}}$. Clearly f is holomorphic everywhere except at $z = 0$ where (d″) fails. On the other hand (a), (b), (c) hold.

We shall give yet another example of a function f which satisfies (a), (b), (c) and

$$\int_{\partial R} f(z)dz = 0 \tag{4.1}$$

for every rectangle R in the plane but has an essential singularity at $z = 0$.

Let $p(z)$ be any non-zero polynomial in z such that $p(0) = 0$. Let $\phi(z) = p(z^2)e^{iz^{-2}}$. It is easy to verify that ϕ satisfes (a), (b), (c) and has an essential singularity at $z = 0$. Let

$$I(\phi, R) = \int_{\partial R} \phi(z)dz$$

for any rectangle R and $S = \{z = x + iy;\ 0 \leqq x,\ y \leqq 1\}$, the unit square. Further

$$I(\phi, R) = 0 \quad \text{or} \quad I(\phi, S) \quad \text{or} \quad -I(\phi, S) \quad \text{for any } R \tag{4.2}$$

depending on R and independent of ϕ.

Clearly $I(\phi, R) = 0$ if R is contained in the second or fourth quadrant since ϕ is continuous on their closure. Further, $I(\phi, R) = 0$ if the closure of R does not contain the origin.

We shall divide the general case into three different cases.

Case (i). Suppose that the origin is a vertex of R and R is contained in the first quadrant. Then $I(\phi, R) = I(\phi, S)$ by the Cauchy integral theorem.

Case (ii). Suppose that the origin is a vertex of R and R is contained in the third quadrant. By a change of variable z to $-z$, we obtain

$$I(\phi, R) = -I(\phi, R')$$

where R' is in the first quadrant, positively oriented and with a vertex at the origin. Thus

$$I(\phi, R) = -I(\phi, S).$$

Case (iii). Suppose that the origin is in the interior of R. Then we can split up R into four rectangles R_1, R_2, R_3 and R_4 located in the first, second, third and fourth quadrant, respectively. From the foregoing considerations we see that $I(\phi, R_2) = I(\phi, R_4) = 0$ because R_2 and R_4 are located in the second and fourth quadrants, respectively. Also $I(\phi, R_1) = I(\phi, S)$ and $I(\phi, R_3) = -I(\phi, S)$ because of cases (i) and (ii). Hence

$$I(\phi, R) = 0.$$

Now, given any rectangle in the plane with sides parallel to the axes, we can split it up into four rectangles each located in a different quadrant and some of them possibly degenerate. The degenerate ones and those located in the even quadrants will contribute zero to the integral. The rest will contribute either zero or $I(\phi, S)$ or $-I(\phi, S)$. Now let

$$\phi_1(z) = z^2 e^{iz^{-2}}, \qquad \phi_2(z) = z^4 e^{iz^{-2}}.$$

We have for any two constants α, β

$$I(\alpha\phi_1 + \beta\phi_2, R) = 0 \quad \text{or} \quad \alpha I(\phi_1, S) + \beta I(\phi_2, S)$$

or its negative. If $I(\phi_1, S) = 0$, we let $\alpha = 1$ and $\beta = 0$. If $I(\phi_2, S) = 0$, we let $\alpha = 0$ and $\beta = 1$. If both $I(\phi_1, S)$ and $I(\phi_2, S)$ are not zero (*and that is the truth*), we let $\alpha = I(\phi_2, S)$, $\beta = -I(\phi_1, S)$ and

$$f = \alpha\phi_1 + \beta\phi_2.$$

This f has all the desired properties.

References

1. R. P. Boas, *Invitation to Complex Analysis*, Random House, New York, 1986.

2. Robert M. Fesq, *Green's formula, linear continuity and Hausdorff measure*, Trans. Am. Math. Soc. **118** (1965), 105–112.

3. E. Goursat, *Sur la définition générale des fonctions analytiques d'aprés Cauchy*, Trans. Am. Math. Soc. **1** (1900), 14–16.

4. H. Looman, *Über die Cauchy-Riemannschen Differential Gleichungen*, Nach. Ges. Wiss. Göttingen, (1923), 97–108.

5. D. Menchoff, *Les Conditions de monogénéité*, Hermann, Paris, 1936, pp. 10–12.

6. P. Montel, *Sur les différentialles totales et les fonctions monogènes*, C. R. Acad. Sci. Paris **156** (1913), 1820–1822.

7. T. Pompeiu, *Sur la continuité des fonctions de variable complexe*, Ann. Fac. Sci. Univ. Toulouse (2) 7 (1905), 264–315.

8. S. Saks, *Théorie de l'intégrale*, Monografje Matematyczne II, Varsovie (1933), 243–245.

9. G. Kh. Sindalovskii, *On the Cauchy–Riemann conditions in the class of functions with summable modulus and some boundary properties of holomorphic functions*, Mat. Sb. **128** (170) No. 3 (1985), 364–382 (Russian).

10. G. P. Tolstoff, *Sur la différentialle totale*, Mat. Sb. **9** (51) (1942), 461–468.

11. G. P. Tolstoff, *Sur les fonctions bornées verifiant les conditions Cauchy–Riemann*, Mat. Sb. **10** (52) (1942), 79–86.